Time discontinuous Galerkin

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Simple parabolic problem

$$
\begin{aligned}
u' - \Delta u &= g, \quad \text{on } \Omega \times (0,T) \\
u &= 0, \quad \text{on } \partial\Omega \times (0,T) \\
u &= u^0 \quad \text{on } \Omega \times \{0\}
\end{aligned}
$$

Find $u \in C^1(0, T, H_0^1(\Omega))$ such that

$$
\begin{aligned}
(u'(t), v) + (\nabla u(t), \nabla v) &= (g(t), v), \quad \forall v \in H_0^1(\Omega), \forall t \in (0,T) \\
u(0) &= u^0,
\end{aligned}
$$

Space discretization

- Let $\mathcal{T}_h = \{K\}$ be a conforming partition of Ω.
- Let $h_K = \mathrm{diam}(K)$ and $h = \max_K h$.
- $X_h = \{v \in H_0^1(\Omega) : v|_K = P^p(K)\}$

Semidiscrete problem

Find $u_h \in C^1(0, T, X_h)$ such that

$$
\begin{aligned}
(u_h'(t), v) + (\nabla u_h(t), \nabla v) &= (g(t), v), \quad \forall v \in X_h, \forall t \in (0, T) \\
(u_h(0), v) &= (u^0, v), \quad \forall v \in X_h
\end{aligned}
$$

Time discretization

- Let $0 = t_0 < \ldots < t_r = T$ be a partition of $[0, T]$ and $I_m = (t_{m-1}, t_m)$.
- Let $\tau_m = |I_m|$ and $\tau = \max_m \tau_m$.
- $X_h^\tau = \{v \in L^2(0, T, X_h) : v|_{I_m} \in P^q(I_m, X_h)\}$
- $v \in X_h^\tau$: $v_\pm^m = v(t_m\pm) = \lim_{t \to t_m\pm} v(t)$, $\{v\}_m = v_+^m - v_-^m$

Time discontinuous Galerkin

Definition

We say that the function $U \in X_h^\tau$ is the approximate solution to the simple parabolic problem if

$$\begin{aligned}
\int_{I_m} (U', v) + (\nabla U, \nabla v)\mathrm{d}t + (\{U\}_{m-1}, v_+^{m-1}) &= \int_{I_m} (g, v)\mathrm{d}t, \\
&\quad \forall v \in X_h^\tau, \ \forall m \\
(U_-^0, v) &= (u^0, v).
\end{aligned}$$

Stability

- Stability: $g = 0$ then $\|U_-^m\| \leq \|U_-^{m-1}\|$

Stability

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- setting $v = 2U$:

$$\|U_-^m\|^2 - \|U_-^{m-1}\|^2 + \|\{U\}_{m-1}\|^2 + 2\int_{I_m} |U|_{H^1(\Omega)}^2 dt = 0$$

TDG solving

- we set orthonormal basis of X_h as $\phi_1, \ldots, \phi_N$ and orthonormal basis of $P^q(I_m)$ as $\varphi_1, \ldots, \varphi_{q+1}$

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- we set orthonormal basis of X_h as $\phi_1, \dots, \phi_N$ and orthonormal basis of $P^q(I_m)$ as $\varphi_1, \dots, \varphi_{q+1}$
- $$A = (a_{i,j}) \qquad a_{ij} = (\nabla \phi_j, \nabla \phi_i),$$
  $$R = (r_{i,j}) \qquad r_{ij} = \int_{I_m} \varphi_i'(t)\varphi_j(t)\,\mathrm{d}t + \varphi_i(t_{m-1})\varphi_j(t_{m-1})$$

TDG solving

$$\begin{pmatrix} r_{1,1}I + \delta_{1,1}A & \cdots & r_{1,q+1}I + \delta_{1,q+1}A \\ \vdots & \ddots & \vdots \\ r_{q+1,1}I + \delta_{q+1,1}A & \cdots & r_{q+1,q+1}I + \delta_{q+1,q+1}A \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_{q+1} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_{q+1} \end{pmatrix}.$$

TDG solving

$$\begin{pmatrix} r_{1,1}I + \delta_{1,1}A & \cdots & r_{1,q+1}I + \delta_{1,q+1}A \\ \vdots & \ddots & \vdots \\ r_{q+1,1}I + \delta_{q+1,1}A & \cdots & r_{q+1,q+1}I + \delta_{q+1,q+1}A \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_{q+1} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_{q+1} \end{pmatrix}.$$

$$AX + XR = B,$$

where $X = [x_1, \dots, x_{q+1}],\ B = [b_1, \dots, b_{q+1}] \in \mathbf{R}^{N,q+1}$

Parabolic projection

- Parabolic projection: $\pi : C([0,T], L^2(\Omega)) \to X_h^\tau$, such that

$$\begin{aligned} \pi u_-^m &= \Pi u^m, \\ \int_{I_m} (u - \pi u, v) dt &= 0, \quad \forall v \in P^{q-1}(I_m, X_h). \end{aligned}$$

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Lemma

Let $u \in W^{q+1,\infty}(I_m, H^{p+1}(\Omega))$. Then

$$\begin{aligned} \sup_{I_m} \|\pi u - u\| &\le C(\tau^{q+1} + h^{p+1}), \\ \sup_{I_m} |\pi u - u|_{H^1(\Omega)} &\le C(\tau^{q+1} + h^{p}). \end{aligned}$$

Error estimates

- We divide the error

$$e(t) = U(t) - u(t) = \underbrace{U(t) - \pi u(t)}_{:=\xi(t)} + \underbrace{\pi u(t) - u(t)}_{:=\eta(t)}$$

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- For η are standard estimates $\|\eta\|_{L^\infty(L^2)}$
- It is sufficient to estimate $\|\xi\|_{L^\infty(L^2)}$ only.

Lemma

Let $u \in C^1([0,T], H_0^1(\Omega))$. *Then*

$$\int_{I_m} ((\pi u - u)', v)dt + (\{\pi u - u\}_{m-1}, v_+^{m-1}) = 0.$$

Error estimates

$$\int_{I_m} (\xi', v) + (\nabla \xi, \nabla v) \mathrm{d}t + (\{\xi\}_{m-1}, v_+^{m-1}) = -\int_{I_m} (\nabla \eta, \nabla v) \mathrm{d}t \\ - \int_{I_m} (\eta', v) \mathrm{d}t - (\{\eta\}_{m-1}, v_+^{m-1})$$

Main result

Theorem

Let $u \in W^{q+1,\infty}(0, T, H^1(\Omega)) \cap W^{1,\infty}(0, T, H^{p+1}(\Omega))$ *be exact solution of simple parabolic problem and* U *be its discrete approximation. Then*

$$\max_{m=1,\dots,r} \|U_-^m - u^m\| \le C(h^p + \tau^{q+1}),$$

where the constant C *depends on* u *and* T*, but is independent of* h *and* τ*.*

Discrete characteristic function

Definition (Chrysafinos, Walkington)

Let $v \in X_h^\tau$ and $y \in [t_{m-1}, t_m]$. Then we say that $\hat{v}_y \in X_h^\tau$ is discrete characteristic function to function v and value y, if

$$\begin{aligned} v_+^{m-1} &= \hat{v}_{y+}^{m-1} \\ \int_{I_m} (\hat{v}_y, w)\mathrm{d}t &= \int_{t_{m-1}}^{y} (v, w)\mathrm{d}t \quad \forall w \in P^{q-1}(I_m, X_h) \end{aligned}$$

Error estimates

Setting $v = 2\xi$:

$$\|\xi_-^m\|^2 - \|\xi_-^{m-1}\|^2 + \|\{\xi\}_{m-1}\|^2 + \int_{I_m} |\xi|^2_{H^1(\Omega)} dt$$
$$\leq \tau_m C \left(\tau^{2q+2} + h^{2p}\right).$$

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Setting $v = 2\hat{\xi}_y$:

$$\sup_{I_m} \|\xi\|^2 - \sup_{I_{m-1}} \|\xi\|^2 \leq C\tau_m \left(\tau^{2q+2} + h^{2p}\right) + \int_{I_m} |\hat{\xi}_y|^2_{H^1(\Omega)} \mathrm{d}t - 2\int_{I_m} (\nabla\xi, \nabla\hat{\xi}_y) \mathrm{d}t \leq C\tau_m \left(\tau^{2q+2} + h^{2p+2}\right) + K \int_{I_m} |\xi|^2_{H^1(\Omega)} \mathrm{d}t$$

Main result

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Let $u \in W^{q+1,\infty}(0,T,H^1(\Omega)) \cap W^{1,\infty}(0,T,H^{p+1}(\Omega))$ *be exact solution of simple parabolic problem and* U *be its discrete approximation. Then*

$$\max_{m=1,\dots,r} \sup_{I_m} \|U-u\| \le C(h^p + \tau^{q+1}),$$

where the constant C *depends on* u *and* T*, but is independent of* h *and* τ*.*

Singularly perturbed problem

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu &= g, \quad \forall x \in (0,1), t \in (0,T), \quad (1)\\
u(0,t) = u(1,t) &= 0, \quad \forall t \in (0,T),\\
u(x,0) &= u^0(x), \quad \forall x \in (0,1),
\end{aligned}
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u(0,t) = u(1,t) &= 0, \quad \forall t \in (0,T),\\
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\end{aligned}
$$

- $g \in L^2((0,1) \times (0,T))$
- $u^0 \in L^2(0,1)$
- $0 < \varepsilon << 1$
- functions b and c are sufficiently smooth with $b(x) > \beta > 0$ $c - \frac{1}{2}\frac{\partial b}{\partial x}(x) \geq c_0 > 0$

Weak formulation

Bilinear form:

$$a(u, v) = \varepsilon(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}) + (b\frac{\partial u}{\partial x} + cu, v)$$

Energy norm:

$$\|v\|_\varepsilon^2 = \|v\|^2 + \varepsilon|v|_{H^1(0,1)}^2$$

$$a(v, v) \geq \min(c_0, 1)\|v\|_\varepsilon^2 \geq 0$$

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Energy norm:

$$\|v\|_{\varepsilon}^2 = \|v\|^2 + \varepsilon|v|^2_{H^1(0,1)}$$

$$a(v,v) \geq \min(c_0, 1)\|v\|_{\varepsilon}^2 \geq 0$$

We say that the function u is the weak solution, if the following conditions are satisfied

$$\begin{aligned} u &\in C^1(0,T,H_0^1(0,1)) \qquad (2) \\ \left(\frac{\partial u(t)}{\partial t}, v\right) + a(u(t),v) &= (g(t),v), \\ &\qquad \forall t \in (0,T), \forall v \in H_0^1(0,1), \\ u(0) &= u^0. \end{aligned}$$

Properties of the exact solution

- Solution has in general boundary layer at $x = 1$.
- Assuming sufficiently compatible data solution has no interior layer.
- Moreover, it is possible to prove

$$\left| \frac{\partial^{k+m} u(x,t)}{\partial^k x \partial^m t} \right| \leq C \left(1 + \frac{1}{\varepsilon^k e^{\beta(1-x)/\varepsilon}} \right) \tag{3}$$

Space discretization

- Shishkin mesh: equidistantly distributed mesh points in intervals $[0, \sigma]$ and $[\sigma, 1]$, $\sigma = \frac{5}{2}\varepsilon \log(N)$.

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- $V_N \subset H_0^1(0,1)$ contains piecewise linear functions

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- Shishkin mesh: equidistantly distributed mesh points in intervals $[0,\sigma]$ and $[\sigma,1]$, $\sigma = \frac{5}{2}\varepsilon \log(N)$.
- N number of mesh points
- $V_N \subset H_0^1(0,1)$ contains piecewise linear functions

Semidiscrete problem: find $u_N \in C^1(0,T,V_N)$ satisfying

$$
\begin{aligned}
\left(\frac{\partial u_N(t)}{\partial t}, v\right) + a(u_N(t), v) &= (g(t), v), \quad \forall v \in V_n, \forall t \in (0,T), \\
(u_N(0), v) &= (u^0, v). \quad \forall v \in V_N
\end{aligned}
$$

Time discretization

- Let $0 = t_0 < \ldots < t_r = T$ be a partition of $[0, T]$ and $I_m = (t_{m-1}, t_m)$.
- Let $\tau_m = |I_m|$ and $\tau = \max_m \tau_m$.
- $V_N^\tau = \{v \in L^2(0, T, V_N) : v|_{I_m} \in P^q(I_m, V_N)\}$
- $v \in V_N^\tau$: $v_\pm^m = v(t_m \pm) = \lim_{t \to t_m \pm} v(t)$, $\{v\}_m = v_+^m - v_-^m$

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Definition

We say that the function $U \in V_N^\tau$ is the approximate solution to the simple parabolic problem if

$$\begin{aligned} \int_{I_m} (U', v) + a(U, v)\mathrm{d}t + (\{U\}_{m-1}, v_+^{m-1}) &= \int_{I_m} (g, v)\mathrm{d}t, \\ &\quad \forall v \in V_N^\tau, \ \forall m \\ (U_-^0, v) &= (u^0, v). \end{aligned}$$

Ritz projection

- Ritz projection: $R : H_0^1(0,1) \to V_N$, such that $a(u,v) = a(Ru,v), \forall v \in V_N$

Ritz projection

- Ritz projection: $R : H_0^1(0,1) \to V_N$, such that $a(u,v) = a(Ru,v)$, $\forall v \in V_N$
- It is possible to prove following estimates:
  $\|Ru - u\|_\varepsilon \leq C(N^{-1} \ln N)$
  $\|Ru - u\| \leq C(N^{-1} \ln N)^2$

Time interpolation

- Let us assume Radau quadrature nodes in interval I_m: $t_{m-1} < t_{m,q} < \ldots < t_{m,0} = t_m$
- $X^\tau = \{v \in L^2(0, T, L^2(0,1)) : v|_{I_m} \in P^q(I_m, L^2(0,1))\}$
- Time projection: $P_\tau : C([0, T], L^2(0,1)) \to X^\tau$, such that

$$P_\tau u(t_{m,i}) \quad = \quad u(t_{m,i})$$

Time interpolation

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- $X^\tau = \{v \in L^2(0, T, L^2(0,1)) : v|_{I_m} \in P^q(I_m, L^2(0,1))\}$
- Time projection: $P_\tau : C([0, T], L^2(0,1)) \to X^\tau$, such that

$$P_\tau u(t_{m,i}) \quad = \quad u(t_{m,i})$$

Then

$$\sup_{I_m} \|P_\tau u - u\| \quad \leq \quad C\tau^{q+1}$$

Space-time projection

- Space-time projection: $\pi = RP_\tau = P_\tau R$

Space-time projection

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- $\sup_{I_m} \|\pi u - u\| \le C(\tau^{q+1} + (N^{-1} \ln N)^2)$

Radau quadrature

- Let $f \in C(0,1)$, we define Radau quadrature:

$$\int_{I_m} f(t)dt \approx Q[f] = \sum_{i=0}^{q} w_i f(t_{m,i})$$

- Radau quadrature has algebraic degree $2q$
- $0 < w_i \leq \tau_m$

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- Radau quadrature has algebraic degree $2q$
- $0 < w_i \leq \tau_m$

If $g \in P^q(I_m, L^2(0,1))$, then we can express the method:

$$Q[(U', v)] + Q[a(U, v)] + (\{U\}_{m-1}, v_+^{m-1}) = Q[(g, v)] \quad \forall v \in V_N^\tau$$

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If $g \in P^q(I_m, L^2(0,1))$, then we can express the method:

$$Q[(U', v)] + Q[a(U, v)] + (\{U\}_{m-1}, v_+^{m-1}) = Q[(g, v)] \quad \forall v \in V_N^\tau$$

For the exact solution we obtain:

$$Q[(u', v)] + Q[a(u, v)] + (\{u\}_{m-1}, v_+^{m-1}) = Q[(g, v)] \quad \forall v \in V_N^\tau$$

We divide the error

$$e(t) = U(t) - u(t) = \underbrace{U(t) - \pi u(t)}_{:=\xi(t)} + \underbrace{\pi u(t) - u(t)}_{:=\eta(t)}$$

Error estimates

We divide the error

$$e(t) = U(t) - u(t) = \underbrace{U(t) - \pi u(t)}_{:=\xi(t)} + \underbrace{\pi u(t) - u(t)}_{:=\eta(t)}$$

$$\int_{I_m} (\xi', v) + a(\xi, v) dt + (\{\xi\}_{m-1}, v_+^{m-1})$$
$$= -Q[(\eta', v)] - (\{\eta\}_{m-1}, v_+^{m-1}) - Q[a(\eta, v)].$$

Error estimates

$$
\begin{aligned}
Q[(\eta', v)] + (\{\eta\}_{m-1}, v_+^{m-1}) &\leq \tau_m C(\tau^{q+1} + (N^{-1} \ln N)^2) \sup_{I_m} \|v\|^2 \\
Q[a(\eta, v)] &= 0
\end{aligned}
$$

Error estimates

$$
\begin{aligned}
Q[(\eta', v)] + (\{\eta\}_{m-1}, v_+^{m-1}) &\leq \tau_m C(\tau^{q+1} + (N^{-1} \ln N)^2) \sup_{I_m} \|v\|^2 \\
Q[a(\eta, v)] &= 0
\end{aligned}
$$

$$
\int_{I_m} (\xi', v) + a(\xi, v) dt + (\{\xi\}_{m-1}, v_+^{m-1})
$$
$$
\leq \tau_m C(\tau^{q+1} + (N^{-1} \ln N)^2) \sup_{I_m} \|v\|^2.
$$

Error estimates

Setting $v = 2\xi$:

$$\|\xi_-^m\|^2 - \|\xi_-^{m-1}\|^2 + \|\{\xi\}_{m-1}\|^2 + 2\min(c_0, 1)\int_{I_m} \|\xi\|_\varepsilon^2 dt$$
$$\leq \tau_m C\left(\tau^{q+1} + (N^{-1}\ln N)^2\right)\sup_{I_m}\|\xi\|.$$

Error estimates

Setting $v = 2\xi$:

$$\|\xi_-^m\|^2 - \|\xi_-^{m-1}\|^2 + \|\{\xi\}_{m-1}\|^2 + 2\min(c_0,1)\int_{I_m}\|\xi\|_\varepsilon^2 dt$$
$$\le \tau_m C\left(\tau^{q+1} + (N^{-1}\ln N)^2\right)\sup_{I_m}\|\xi\|.$$

[Akrivis, Makridakis, 2004]: setting $v = 2\tilde{\xi}$, where $\tilde{\xi}$ is interpolation of $\frac{\tau_m}{t-t_{m-1}}\xi(t)$:

$$\sup_{I_m}\|\xi\|^2 \le C\left(\|\xi_-^m\|^2 + \frac{1}{\tau_m}\int_{I_m}\|\tilde{\xi}\|^2 dt + 2\min(c_0,1)\int_{I_m}\|\xi\|_\varepsilon^2 dt\right)$$
$$\le C\left((\xi_-^{m-1}, \tilde{\xi}_+^{m-1}) + C\left(\tau^{q+1} + (N^{-1}\ln N)^2\right)\sup_{I_m}\|\xi\|\right)$$
$$\le C\left(\|\xi_-^{m-1}\|^2 + \tau^{2q+2} + (N^{-1}\ln N)^4\right) + \frac{1}{2}\sup_{I_m}\|\xi\|^2$$

Thank you for your attention.