The key results obtained by using (not my) simple ideas

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Method of trajectories

- Abstract setting: Have one-parameterized semigroup (solution operator) $S_t : V \to V$ for all $t \geq 0$, i.e.,

$$S_0 = I; \qquad S_{t+s} = S_t S_s$$

- what happens as $t \to \infty$? Define attractor $\mathcal{A} \subset V$ a s a compact set having the properties

$$S_t(\mathcal{A}) = \mathcal{A}; \qquad \inf_{u \in \mathcal{A}} \|S_t u_0 - u\|_V \overset{t\to\infty}{\to} 0$$

- interesting questions: existence of $\mathcal{A}$ and its fractal/Hausdorff dimension
- "ideal method" for PDE's seems to be Lyapunov exponents, but work only if one has enough regularity (which is usually not the case, sometimes in principle)

Method of trajectories II

The first interest was to apply it to flows of incompressible homogeneous fluids with pressure depended viscosities $\Longrightarrow$ after some years of improving we simplified the way how to estimates the dimension & to obtain results comparable with Lyapunov exponents.

- fluids

$$\boxed{\begin{aligned} \operatorname{div} \mathbf{v} &= 0, \\ \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} &= -\nabla p + \mathbf{f} \end{aligned}}$$

  with $\mathbf{S} \sim (\nu(p) + |\mathbf{D}(\mathbf{v})|^2)^{\frac{r-2}{2}} \mathbf{D}(\mathbf{v})$

- hyperbolic equation

$$\boxed{u_{tt} + g(u_t) + Au + f(u) = 0}$$

Method of trajectories III

Málek & Nečas idea:
$S_t(u_0) = u(t)$ for any $u \in H$ ($H = L^2_{\mathrm{div}}(\Omega)^d$ for fluids and $H = L^2 \times W^{1,2}$ for hyperbolic eq.)
Instead of point behavior, consider the ℓ trajectory, i.e. $\chi_\ell(t) \in L^2(0,\ell;H)$ is defined as

$$[\chi_\ell(t)](s) := u(t+s) \qquad s \in (0,\ell)$$

and define another semigroup L^ℓ_t as

$$L^\ell_t(\chi_\ell(0)) := \chi_\ell(t)$$

- advantages: compactness for nothing (Aubin-Lions), almost immediate estimate for attractor to L^ℓ, easy identifications with the original attractor
- disadvantage: everything depends on the length ℓ-optimization of ℓ, optimization of the use of a kind of Aubin-Lions

Navier-Stokes-Fourier system

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\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} &= -\nabla p + \mathbf{f}, \\
e_t + \operatorname{div}(e\mathbf{v}) - \operatorname{div}(\kappa(e)|\nabla e|^{s-2}\nabla e) &= \mathbf{S} \cdot \mathbf{D}(\mathbf{v})
\end{aligned}
$$

with $\mathbf{S} \sim \nu(e)(\nu_1(p) + |\mathbf{D}|^2)^{\frac{r-2}{2}}\mathbf{D}(\mathbf{v})$ or $\mathbf{F}(\mathbf{S}, \mathbf{D}) = \mathbf{0}$

Goal:

- Develop general existence theory whenever a priori estimates imply compactness of convective term

Problems:

- existence of a pressure as an integrable function
- what to do with the VERY BAD term on the right hand side of the heat equation
- what to do in case $\mathbf{S}$ is not continuous function of $\mathbf{D}$
- what to do if you cannot simply apply monotone operator theory, i.e., if you cannot test by solution

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Navier-Stokes-Fourier system II

Solutions:

- Pressure: consider Navier slip bc., i.e.,

$$\mathbf{v} \cdot \mathbf{n} = 0, \quad (\mathbf{S}\mathbf{n})_{\tau} = -\alpha \mathbf{v} \qquad \text{on } \partial\Omega$$

  Justification: for homogeneous Dirichlet, existence of pressure is not clear; Navier bc. is a good approximation of Dirichlet bc. in case of oscillating boundaries, it is used in praxis for large stresses

- BAD TERM: do not consider the heat equation but the equation for the global energy $E := \frac{1}{2}|\mathbf{v}|^2 + e$, i.e.,

$$\boxed{E_t + \operatorname{div}(\mathbf{v}(E+p)) - \operatorname{div}(\mathbf{S}\mathbf{v} - \kappa(e)|\nabla e|^{s-2}\nabla e = \mathbf{f} \cdot \mathbf{v}}$$

  Justification: it is in fact the correct setting (the first law of thermodynamics); no presence of BAD TERM except the pressure; but we know what to do with the pressure

- Implicit relations: Think implicitly; replace operators by maximal monotone graphs; do not rely on point-wise convergence-weak is enough
- Non-applicability monotone operator theory: Use an approximation of the solution as a test function; the approximation must be closed to the original one; method of L^{∞} and/or Lipschitz aproximation of Bochner functions.

Navier-Stokes-Fourier system III

Results:

- NSF- existence of weak solution for $s = 2$ and $r \in (\frac{9}{5}, 2)$ (or $r > \frac{9}{5}$)
- theory for general nonlinear convection diffusion equation with right hand side being a measure (general bc), i.e.,

$$\boxed{e_t + \mathrm{div}(e\mathbf{v}) - \mathrm{div}(\kappa(e)|\nabla e|^{s-2}\nabla e) = \mu}$$

  where $\mu \in \mathcal{M}((0,T) \times \Omega)$ and $\mathbf{v} \in L^1(0,T; L^1(\Omega))$ being divergence free with zero trace (even more general convection is allowed).
  sequential compactness of ∇e for any data; **existence of a weak solution (and entropy solution)** for all $s > 1$ provided that the formulation are meaningful; **uniqueness of entropy solution** if $\mathbf{v} \in L^{q'}(0,T; L^{q'}(\Omega))$ and $\mu \in L^1$.
- Implicit theory (no temperature): identify $F(\mathbf{S}, \mathbf{D}) = \mathbf{0}$ with the graph $\mathcal{A} \subset \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d}$ that is maximal monotone (Bingam model etc.) and consider a general response

$$\forall (\mathbf{S}, \mathbf{D}) \in \mathcal{A}: \quad \mathbf{S} \cdot \mathbf{D} \geq \alpha(\psi(|\mathbf{D}|) + \psi^*(|\mathbf{S}|)) - C$$

  existence of a weak solution for any bc whenever $\psi(s) \geq s^{\frac{6}{5}+\delta} - C$
- **Maximal L^2 regularity** for $r = s = 2$ without convective terms in any d for special ν's

Monotone operator theory

- $\mathbf{S}(x, \mathbf{D}(\mathbf{v}))$ is Carathéodory

$$(\mathbf{S}(x, \mathbf{D}_1) - \mathbf{S}(x, \mathbf{D}_2)) \cdot (\mathbf{D}_1 - \mathbf{D}_2) \geq \alpha |\mathbf{D}_1 - \mathbf{D}_2|^2,$$
$$|\mathbf{S}(x, \mathbf{D}_1) - \mathbf{S}(x, \mathbf{D}_2)| \leq L |\mathbf{D}_1 - \mathbf{D}_2|$$

- Solve

$$\boxed{-\operatorname{div} \mathbf{S}(x, \mathbf{D}(\mathbf{v})) = \operatorname{div} \mathbf{F}}$$

- $\mathbf{F} \in L^2 \implies$ there is a unique weak solution $\mathbf{u} \in W^{1,2}$
- **???** for $p \in (1, 2)$: $\mathbf{F} \in L^p \implies$ there is a unique weak solution $\mathbf{u} \in W^{1,p}$
- No in general: counterexample due to Serrin (linear operator), but it seemed that for some range (depending on α, L) $p \in (p_0, 2]$ it should be true

Monotone operator theory

- Test by Lipschitz approximation $\mathbf{v}_\lambda$; $\mathbf{v}_\lambda(x) = \mathbf{v}(x)$ if $|\nabla \mathbf{v}(x)| \le \lambda$ and $|\nabla \mathbf{v}_\lambda| \le C\lambda$ to get
$$\int_{|\nabla \mathbf{v}| \le \lambda} |\nabla \mathbf{v}|^2 \le C\left(\lambda \int_{|\nabla \mathbf{v}| \ge \lambda} |\nabla \mathbf{v}| + |\mathbf{F}| + \int_{|\nabla \mathbf{v}| \le \lambda} |\mathbf{F}||\nabla \mathbf{v}|\right)$$
- Multiply by λ^{p-3} and integrate over $(0, \infty)$ - use Fubini

$$\boxed{\frac{1}{2-p}\int |\nabla \mathbf{v}|^p \le C\left(\frac{1}{p-1}\int |\nabla \mathbf{v}|^p + |\mathbf{F}||\nabla \mathbf{v}|^{p-1} + \frac{1}{2-p}\int |\mathbf{F}||\nabla \mathbf{v}|^{p-1}\right)}$$

- if $C|p-2|/(p-1) < 1$ we are done :)

Scalar hyperbolic conservation laws I

Cauchy problem:

$$\boxed{\begin{aligned} u_t - \operatorname{div} \mathbf{f}(u) = 0 &\quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d \\ u(0) = u_0 &\quad \text{in } \mathbb{R}^d \end{aligned}}$$

with $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.
Uniqueness criteria (Kružkov entropy solution): for all $k \in \mathbb{R}$ there holds

$$\boxed{|u-k|_t - \operatorname{div}\left(\operatorname{sign}(u-k)(\mathbf{f}(u) - \mathbf{f}(k)\right) \leq 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d}$$

- Kružkov (later Szepeschy) uniqueness if $\mathbf{f} \in \mathcal{C}^{0,\alpha}$ with $\alpha > \frac{d-1}{d}$.
- what to do if $\mathbf{f}$ is discontinuous?
- if $\mathbf{f}$ is not enough Hölder continuous at 0-no uniqueness (Kružkov)
- only few results and only 1D - nonnatural definitions etc..

Scalar hyperbolic conservation laws II

Think implicitly!
For any jump continuous $\mathbf{f}$ we can find continuous functions $(\mathbf{A}, U)$ such that U is non-decreasing and

$$\boxed{\mathbf{A}(s) \in \mathbf{f}(U(s)) \qquad \mathbf{A} \text{ is linear on intervals where } U \text{ is constant}}$$

In case that U is strictly monotone (hence $\mathbf{f}$ is continuous), one can invert it and define

$$\boxed{g(t,x) := U^{-1}(u(t,x)) \iff U(g(t,x)) = u(t,x)}$$

which solves
Cauchy problem II:

$$\boxed{\begin{aligned} U(g)_t - \operatorname{div} \mathbf{A}(g) &= 0 && \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ U(g(0,\cdot)) &= u_0(\cdot) && \text{in } \mathbb{R}^d \end{aligned}}$$

Entropy solution for u is equivalent to the notation: for all $k \in \mathbb{R}$ there holds

$$\boxed{|U(g) - U(k)|_t - \operatorname{div} \left(\operatorname{sign}(g-k)(\mathbf{A}(g) - \mathbf{A}(k))\right) \le 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d}$$

Scalar hyperbolic conservation laws III

- if **f** is $\mathcal{C}^{0,\alpha}$ Hölder continuous **at 0** (with the same α as before) and **f** is **jump continuous** then we have existence of $g, \mathbf{A}, U$ fulfilling the Cauchy problem II. Moreover, the function u

$$u(t,x) := U(g(t,x))$$

does not depend on the choice of $(\mathbf{A}, U)$ and is "stable", i.e., if data converges then solution converges
- extension to the case $\mathbf{f}(x,u)$ - big deal
- done if $\mathbf{f}(x,u) = \tilde{\mathbf{f}}(\theta(x,u))$ with $\tilde{\mathbf{f}}$ being jump continuous and θ strictly increasing w.r.t u and measurable in x.
- Observation!!! - solution depends on the choice of $\tilde{\mathbf{f}}$ and θ!!!, but we are still able to identify EXACTLY a class of parametrization that leads to the same solution

New regularity techniques for elliptic systems I

Games theory:

$$\boxed{-\triangle \mathbf{u} = \mathbf{F}(x, \nabla \mathbf{u}) \sim K + |\nabla \mathbf{u}|^2}$$

- if **F** has structure that allows you to get $W^{1,2}$ estimates we developed a theory how to get regularity in any dimension (upper and lower semicontinuity method)

Nonlinear principle part:

$$\boxed{\text{Minimize } J(\mathbf{u}) := \int_\Omega F(x, \mathbf{u}, \nabla \mathbf{u}) - \mathbf{f} \cdot \mathbf{u}}$$

- new theory based on a "new conservation law" - Noether equation-full stationary point, etc.

New regularity techniques for elliptic systems II

Have $F(x, \mathbf{u}, \boldsymbol{\eta}) \sim |\boldsymbol{\eta}|^p$

Nonlinear principle part:

$$\boxed{\text{Minimize } J(\mathbf{u}) := \int_\Omega F(x, \mathbf{u}, \nabla \mathbf{u}) - \mathbf{f} \cdot \mathbf{u}}$$

Euler-Lagrange equations:

$$\boxed{-\operatorname{div} F_{\boldsymbol{\eta}}(x, \mathbf{u}, \nabla \mathbf{u}) + F_{\mathbf{u}}(x, \mathbf{u}, \nabla \mathbf{u}) = \mathbf{f}}$$

- F independent of $\mathbf{u}$ and $p = 2$ and uniformly convex, famous Hilbert problem
- $\mathbf{u}$- scalar - de Giorgi & Nash - regularity
- $\mathbf{u}$-vector - Nečas & Šverák - non-lipschitz & unbounded solution
- F independent of $\mathbf{u}$, $p \in (1, \infty)$ and $F(\nabla \mathbf{u}) = \tilde{F}(|\nabla \mathbf{u}|)$ and p-"convexity" - regular solutions due to Uhlenbeck
- NOTHING BETWEEN

New regularity techniques for elliptic systems III

Noether Equation:

$$-\partial_{x_j}\left(F_{\eta_j^\alpha}(\mathbf{u},\nabla\mathbf{u})\partial_{x_k}\mathbf{u}^\alpha\right)+\partial_{x_k}F(\mathbf{u},\nabla\mathbf{u})=\mathbf{f}^\alpha\partial_{x_k}\mathbf{u}^\alpha \qquad \forall k=1,\dots,d$$

How to get it

- either multiply αth- equation by $\partial_{x_k}\mathbf{u}^\alpha$ and do manipulations - need smoother solution $W^{1,p+1}\cap W^{2,\frac{p+1}{2}}$
- minimize "formally" with respect to inner variable, i.e., define $\mathbf{v}(x):=\mathbf{u}(x+t\psi(x))$ use
$$J(\mathbf{u})\le J(\mathbf{v})$$
and let $t\to 0_+$
- known for $\mathcal{C}^1$ solution, first novelty is that it holds for any $W^{1,p}$ minimizer
- test Noether by $x|x|^{d-p}$ times cut-off to get
- $$\int_{ball}\frac{F_{\eta_j}\partial_{x_k}\mathbf{u}x_jx_k}{|x|^{d-p+2}}\le OK+\int_{ball\ \ with\ \ hole}\frac{|\nabla\mathbf{u}|^p}{|x|^{d-p}}$$

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New regularity techniques for elliptic systems IV

Splitting condition:

$$F \sim |\nabla \mathbf{u}|^q |\nabla \mathbf{u}|_1^s, \quad |\nabla \mathbf{u}|_1^2 := A^{\alpha\beta} \nabla u^\alpha \cdot \nabla u^\beta$$

gives

$$\int_{ball} \frac{|\nabla \mathbf{u}|^{p-2} |\nabla \mathbf{u} \cdot x|^2}{|x|^{d-p+2}} \le OK + \int_{ball\ \ with\ \ hole} \frac{|\nabla \mathbf{u}|^p}{|x|^{d-p}}$$

- F independent of $\mathbf{u}$: use Caciopoli inequality BUT with structure and then improved version of hole-filling
- F depends of $\mathbf{u}$ and satisfies the one-sided condition, $p = 2$: Splitting gives more - $|\int_{B_R} \mathbf{u}| \le CR^d |\ln R|^{\frac{1}{2}}$; improve anisotropic hole-filling to show $\mathbf{u} \in VMO$, the rest is standard, i.e., use theory for F being $\mathbf{u}$-independent

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