Orthogonality in L^p spaces

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Orthogonal functions in L^2

Consider space $L^2[0,1]$ and system of functions f_j. The system is orthonormal if and only if for any $\{a_j\} \in l^2$

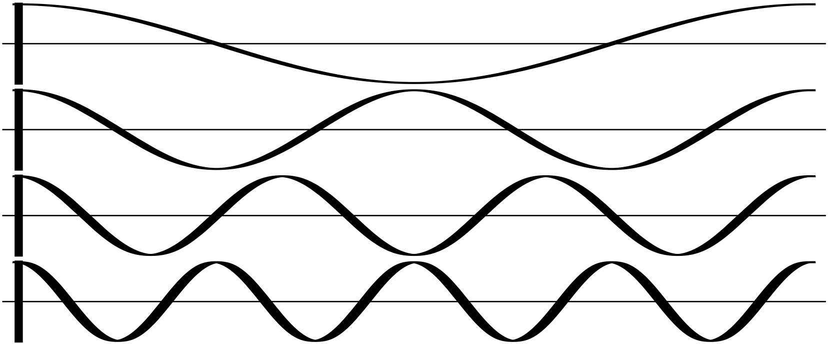
$$\|\sum_j a_j f_j\|_{L^2} = \|a_j\|_{l^2},$$

or orthogonal iff

$$\|\sum_j a_j f_j\|_{L^2} = \left\| \left(\sum_j |a_j|^2 |f_j|^2 \right)^{1/2} \right\|_{L^2}.$$

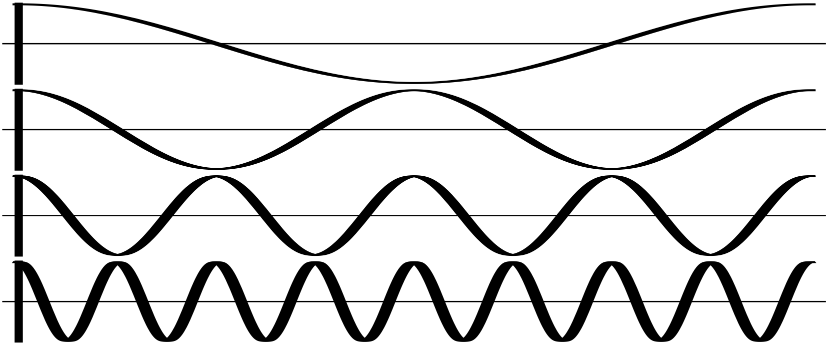
Complex exponentials

$$\phi_j(x) = \mathrm{Exp}(2\pi \mathbf{i} j \cdot x)$$



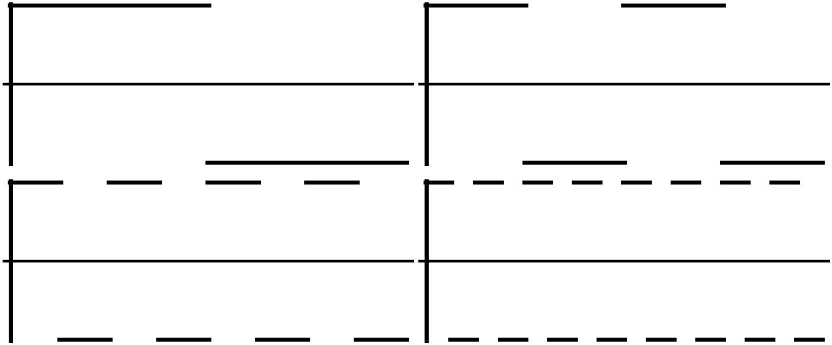
Dyadic exponentials

$$\psi_j(x) = \mathrm{Exp}(2\pi \mathbf{i} 2^j \cdot x)$$



Square wave - Rademachers

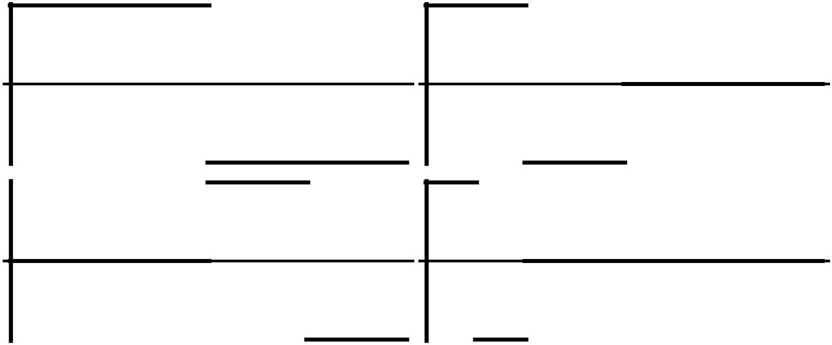
$$r_j(x) = \mathrm{Sgn}(\mathrm{Sin}(2\pi 2^j \cdot x))$$



Haar functions - square wavelets

$$h_{i,j}(x) = |I|^{-1/2}(\chi_{\text{left}(I)} - \chi_{\text{right}(I)})$$

I is dyadic interval, $|I| = 2^{-i}$, jth



Almost orthogonal functions in L^p

On $L^p[0,1]$ with f_j. The system is almost orthonormal if and only if for any $\{a_j\} \in l^2$

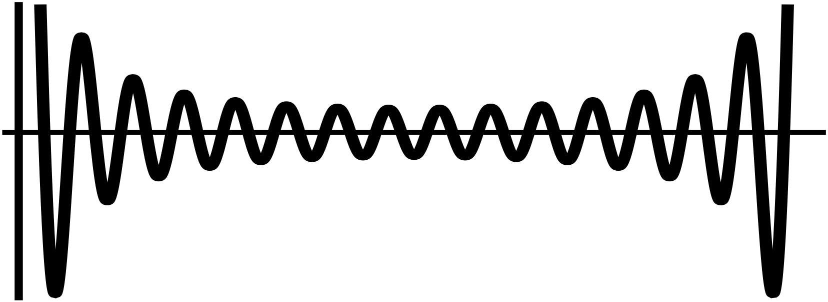
$$\|\sum_j a_j f_j\|_{L^p} \approx \|a_j\|_{l^2},$$

or almost orthogonal

$$\|\sum_j a_j f_j\|_{L^p} \approx \left\| \left(\sum_j |a_j|^2 |f_j|^2 \right)^{1/2} \right\|_{L^p}.$$

Complex exponentials in L^p

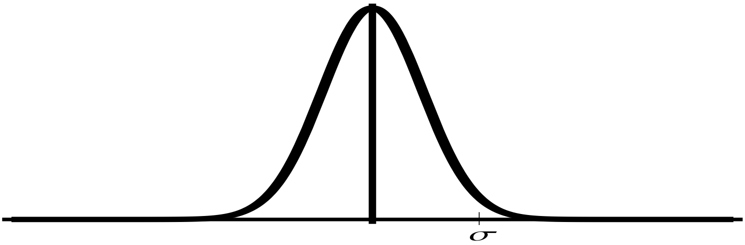
$$\Big|\sum_{|j|\leq N}\phi_j(x)\Big| \approx \min\Big(N\chi_{[0,1/N]}, \frac{1}{|x|}\Big).$$



L^p norm about $N^{\frac{p-1}{p}}$ $(p > 1)$, should be $\approx N^{1/2}$

Rademachers in L^p

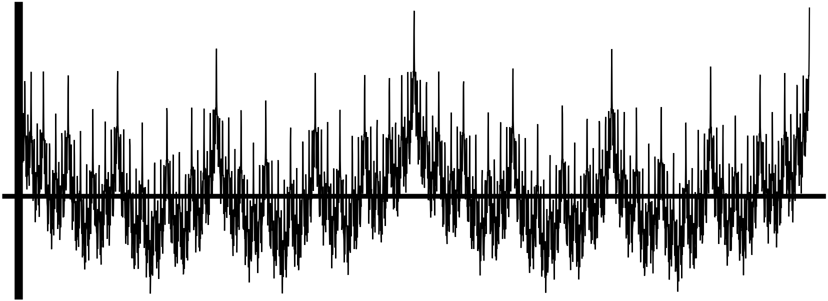
$\sum_{|j|\leq N} r_j(x)$ is a sum of iid random variables, finite variance, central limit theorem gives Gaussian distribution



$\sigma \approx N^{1/2}$, $0 < p < \infty$ L^p (quasi)norm $\approx N^{1/2}$ (Khintchine inequality)

Dyadic complex exponentials in L^p

$\sum_{j=1}^{N} \psi_j(x)$ "lacunary" series, behaves almost like independent variables



$0 < p < \infty$ L^p (quasi)norm $\approx N^{1/2}$

Haar wavelets in L^p

$$\sum_{j=1}^{2^i} a_j 2^{i/2-i/p} h_{j,i}(x)$$

wavelets in the same generations have disjoint supports, therefore L^p norm comes to $\|a\|_{l^p}$

the floors sum with the square, the mix is useless

Square function

New operator square function

$$a_{i,j} = \langle f, h_{i,j} \rangle$$

$$S(f) = \left(\sum_{i,j} |a_{i,j} h_{j,i}|^2(x) \right)^{1/2}.$$

Deep theorem (Littlewood - Paley)

$$\|f\|_p \approx \|Sf\|_p$$

for $1 < p < \infty$.

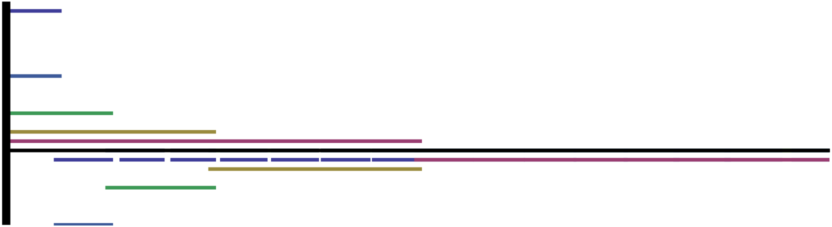
L^1 failure

f is Dirac function

$$a_{i,j} = \langle f, h_{i,j} \rangle$$

$$S_N(f) = \left(\sum_{i<N,j} |a_{i,j} h_{j,i}|^2(x) \right)^{1/2} .$$

$$\|S_N f\|_1 \approx N$$



Maximal function

The operator S has better interplay with

$$Mf(x) = \sup_{x \in I \,\text{dyadic}} \left| \frac{1}{|I|} \int_I f(x) \right| .$$

not only

$$\|Sf\|_p \approx \|Mf\|_p$$

$0 < p < \infty$ but also Central limit theorem (for martingales)

$$|\{Mf > 2\lambda; Sf \leq \epsilon\lambda\}| \leq Ce^{-c/\epsilon^2} |\{Mf > \lambda\}|$$

Wavepacks

If smoothness is of concern (Sobolev spaces), decompositions into packs of smooth waves work better

$$f = \sum_j a_j \phi_j$$

define

$$D_k f = \sum_{2^k \le |j| < 2^{k+1}} a_j \phi_j$$

($D_{-1} = a_0 \phi_0$) we again have square function

$$\widetilde{S} f = \left(\sum_k |D_k f|^2 \right)^{1/2}$$

and

$$\|\widetilde{S} f\|_p \approx \|Mf\|_p$$

for $1 < p < \infty$

Wavepacks - Example

Sawtooth wave, wavepacks $(3^j, 3^{j+1}]$